

# Normal Modes of the B=4 Skyrme Soliton

Chris Barnes\*, Kim Baskerville†, and Neil Turok‡

\*Joseph Henry Laboratory, Princeton University, Princeton, NJ, USA 08540

†Physics Department, University of Wales Swansea, Singleton Park, Swansea SA2 8PP

‡DAMTP, Silver St, Cambridge, CB3 9EW, U.K.

(February 1, 2008)

The Skyrme model of nuclear physics requires quantisation if it is to match observed nuclear properties. A simple technique is used to find the normal mode spectrum of the baryon number  $B = 4$  Skyrme soliton, representing the  $\alpha$  particle. We find sixteen vibrational modes and classify them under the cubic symmetry group  $O_h$  of the static solution. The spectrum possesses a remarkable structure, with the lowest energy modes lying in those representations expected from an approximate correspondence between Skyrmions and BPS monopoles. The next mode up is the ‘breather’, and above that are higher multipole breathing modes.

## I. INTRODUCTION

In the Skyrme model of nuclear physics, both pions and nucleons are represented by a single scalar  $SU(2)$  group valued field,  $U(x)$ . Pions occur as field quanta, while baryons are instead represented as topological solitons. The classical Skyrme theory, with a simple quantisation of the spin and isospin collective modes, provides a description of nucleons and the  $\Delta$  resonance in modest agreement with experiment [1,2].

Applying the Skyrme model to larger nuclei and to nuclear matter is an even more interesting proposition, since with no additional free parameters one could compare the theory with the binding energies and gamma ray spectra of all nuclei. There has been progress in understanding the structure of Skyrme multisolitons [3], [4], but it is clear that unless quantum fluctuations about the static solutions are included, there is little chance of success. For real nuclei the relative kinetic energies of the nucleons in the ground state are large, so a quantisation at least of a number of degrees of freedom equal to the number of nucleon coordinates is essential. Note that the simple collective coordinate quantisations of spin and isospin [1,2] include effects of order  $\hbar^2$ , while ignoring effects of order  $\hbar$ .

Recently there has been some progress in quantisation, based upon Manton’s notion of representing low energy solitonic excitations as motion on a finite dimensional space of moduli. A study of the deuteron by Leese et. al. [5] using an instanton approximation for the field configurations, gave encouraging agreement with experimental properties, but only included two of the four expected vibrational modes of the deuteron. Walet [6] extended this treatment to estimate all vibrational frequencies for the deuteron and triton, again using the instanton approximation.

In this Letter we follow a different track. We directly compute the low energy normal modes, finding their frequencies and the representations they lie in under the static soliton’s symmetry group. The frequencies and representations provide a coordinate-independent

description of the configuration space around the static solution; in the harmonic approximation they determine the quantum vibrational energy levels. Our results provide new insight to the moduli space approach, since the representations for the lowest frequency modes turn out to be just those expected from a recently understood approximate correspondence between Skyrmions and BPS monopoles [8]. Furthermore, the success of these calculations allows one to contemplate going beyond the moduli space approximation and performing a full semiclassical quantisation of the field theory. This is an attractive goal, since the Skyrme theory could then be incorporated in the framework of chiral effective Lagrangians [10], allowing a unified treatment of mesons, baryons and higher nuclei.

## II. METHOD

Since the  $SU(2)$  manifold is a 3-sphere, we represent it in terms of a scalar field  $\phi \in \mathbf{R}^4$ , with  $\phi^a \phi^a = 1$ . In terms of this field, the Skyrme Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi + \omega_\pi^2 \phi^1 + \lambda (\phi \cdot \phi - 1) + \frac{1}{4} \{ (\partial_\mu \phi \cdot \partial_\nu \phi) (\partial^\mu \phi \cdot \partial^\nu \phi) - (\partial_\mu \phi \cdot \partial^\mu \phi) (\partial_\nu \phi \cdot \partial^\nu \phi) \} \quad (1)$$

with  $\lambda$  a Lagrange multiplier field. Here, length and time are in units of  $\frac{2}{eF_\pi}$ , energy in units of  $\frac{F_\pi}{2e}$ . In these rescaled units, the only remaining parameter is  $\omega_\pi = \frac{2m_\pi}{eF_\pi}$ , the oscillation frequency for the homogenous pion field. For the most part we have adopted the ‘standard’ value [2] of  $\omega_\pi = .526$ , although we have also performed calculations at twice this value.

The mixed space and time derivative terms in the Skyrme Lagrangian make numerical solution in general difficult. Isolating the time derivative terms one has

$$\mathcal{L} = \frac{1}{2} \dot{\phi}^a K^{ab} (\partial_i \phi) \dot{\phi}^b - V(\phi, \partial_i \phi) + \lambda (\phi \cdot \phi - 1), \quad (2)$$

where  $K^{ab} = \delta^{ab} (1 + (\partial_i \phi)^2) - \partial_i \phi^a \partial_i \phi^b$  is a local inertia matrix, and  $V(\phi, \partial_i \phi)$  is the potential. In general,  $K^{ab}$

is time dependent, but for small perturbations around a static solution  $\phi_{\text{st}}(\mathbf{x})$  we write  $\phi(\mathbf{x}, t) = \phi_{\text{st}}(\mathbf{x}) + \epsilon(\mathbf{x}, t)$ ,  $\epsilon \ll 1$  and to second order in  $\epsilon$  the Lagrangian is

$$\mathcal{L} = \frac{1}{2} \epsilon^a K^{ab} (\partial \phi_{\text{st}}) \epsilon^b - V(\phi, \partial_i \phi) + \lambda(\phi \cdot \phi - 1) \quad (3)$$

This Lagrangian leads to classical field equations:

$$K^{ab}(\partial \phi_{\text{st}}) \ddot{\phi}^a = \partial_i \left( \frac{\partial V}{\partial \phi^b_{,i}} \right) - \frac{\partial V}{\partial \phi^b} + \lambda \phi^b. \quad (4)$$

where the matrix  $K^{ab}(\mathbf{x})$  is taken to be its value at the static classical solution. Equation (4) closely approximates the Skyrme equations for fields near a static classical solution, precisely the desired regime for studying soliton normal modes.

In order to numerically solve the field equations, we discretise the action, (3), using a diagonal differencing scheme for the four spatial derivative terms, achieving a high degree of locality and second order accuracy in both spatial and time steps. The numerical code conserves energy and baryon number to within 1 part in  $10^5$  over the course of extremely long (50k timestep) runs. Periodic boundary conditions are used. We first create the appropriate minimal energy static solution by straightforward time evolution from four-Skyrmion initial conditions. A simple relaxation procedure sets the field momenta zero each time the kinetic energy reaches a maximum. We find the fields rapidly converge on the minimum energy configuration.  $K^{ab}$  is set equal to  $\delta^{ab}$  in this part of the calculation, since this does not affect the final static solution.

Next we slightly perturb the fields and evolve them forward again, but now using the full inertia matrix  $K^{ab}(\partial \phi_{\text{st}})$ . The evolving field is

$$\phi(\mathbf{x}, t) = \phi_{\text{st}}(\mathbf{x}) + \sum_{\text{modes}} \epsilon_n \delta_n(\mathbf{x}) \cos(\omega_n t) + \mathcal{O}(\epsilon^2) \quad (5)$$

where the functions  $\delta_n(\mathbf{x}) \in \mathbf{R}^4$ , obeying  $\delta_n(\mathbf{x}) \cdot \phi_{\text{st}}(\mathbf{x}) = 0$ , are the normal modes, each excited with amplitude  $\epsilon_n$ . The normal mode frequencies  $\omega_n$  are found by Fourier transforming  $\phi(\mathbf{x}, t)$  with respect to time at any point in the box, and plotting the resulting power spectrum.

The space of perturbations has a useful inner product

$$\langle \delta_1 | \delta_2 \rangle = \int_{\text{box}} \delta_1^a(\mathbf{x}) K^{ab}(\mathbf{x}) \delta_2^b(\mathbf{x}) d^3 \mathbf{x}, \quad (6)$$

which is zero for normal modes  $\delta_1$  and  $\delta_2$  if  $\omega_1 \neq \omega_2$ . The inner product allows one to determine the degeneracies of the normal mode frequencies and the representations in which the modes transform under the soliton's symmetry group.

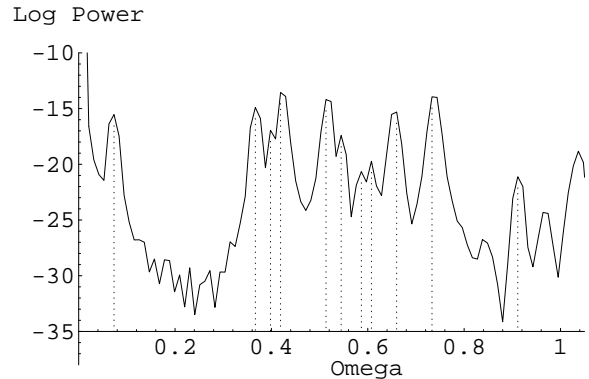


FIG. 1. Fourier power spectrum for perturbations around a B=4 soliton. Frequency is in Skyrme units, power scale is arbitrary. Note that the frequency of homogeneous pion oscillations is  $\omega_\pi = .526$  in these units.

### III. RESULTS FOR THE B=4 SOLITON

We have applied this technique to the case of the Skyrme soliton with baryon number four: the  $\alpha$  particle. The  $\alpha$  particle provides an especially simple case for quantisation, because the ground state possesses zero angular momentum and isospin. The static soliton has cubic symmetry; its energy and baryon number density concentrate along the edges of a cube [4]. The full 48 dimensional cubic group of symmetries  $O_h$  (for notation see [11]) is generated by 90 degree and 120 degree rotations, and parity  $I$ . After an appropriate global isospin rotation, the action of these group elements on spatial coordinates and pion fields ( $\phi^a = (\sigma, \vec{\pi})$ ) is as follows [7]:

$$\begin{aligned} C_4 : & (\pi^1, \pi^2, \pi^3)(x, y, z) \rightarrow (-\pi^2, -\pi^1, -\pi^3)(-y, x, z) \\ C_3 : & (\pi^1, \pi^2, \pi^3)(x, y, z) \rightarrow (\pi^2, \pi^3, \pi^1)(y, z, x) \\ I : & (\pi^1, \pi^2, \pi^3)(x, y, z) \rightarrow (\tilde{\pi}^1, \tilde{\pi}^2, \tilde{\pi}^3)(-x, -y, -z) \end{aligned}$$

with  $\tilde{\pi}^1 = \frac{1}{3}(\pi^1 - 2\pi^2 - 2\pi^3)$  etc. From this it is straightforward to check that a homogeneous pion field falls into the two dimensional representation  $E^+$  and the one dimensional representation  $A_2^-$ , where superscripts indicate parity.

A typical power spectrum of the perturbations is shown in Figure 1. Spectra at different sites and for different field components show the same peaks, but with differing heights. Once the normal mode frequencies  $\omega_n$  are identified, maps of the normal modes  $\delta_n(\mathbf{x})$  may be constructed by performing discrete Fourier sums on each component of the field as it evolves. For degenerate modes, each set of perturbed initial conditions gives a different linear combination  $\sum_i \epsilon_i \delta_i(\mathbf{x})$  of modes with the same frequency. Other linear combinations may also be produced by applying the symmetries of the static soliton  $\mathcal{S}_1, \mathcal{S}_2, \dots$  to a given mode  $\delta$ . The degeneracy of a given frequency is found by computing the rank of the matrix of inner products between the different linear combinations of degenerate modes produced in these ways. Once the degeneracy

is determined, a complete orthonormal basis of modes with this frequency can be constructed. The character of any  $O_h$  group element can then be computed as a trace. These characters were within  $\pm 0.001$  of an integer value, and interpretation was unambiguous.

Each peak in Figure 1 marks the frequency of a normal mode. The lowest peak (at  $\omega = .07$ ) is the rotational zero mode, shifted to nonzero frequency by finite size effects, effectively through interactions with image solitons one box length away. There are also several peaks corresponding to relatively delocalised modes which we interpret as pion radiation. Two (at  $\omega = .545, .587$ ) correspond to the lowest radiation modes, homogeneous away from the soliton, whose threefold degeneracy is split into  $E^+ + A_2^-$  by the presence of the soliton. The first radiation mode at nonzero wavenumber is at  $\omega = .908$ . The remaining modes are the true vibrational excitations of the  $\alpha$  particle. Somewhat fortuitously, the box size was small enough that the lowest inhomogeneous radiation mode has a frequency above the highest vibrational mode.

Four widely separated  $B = 1$  Skyrmons have 24 zero modes, corresponding to 3 translations and 3 isorotations each. As they combine to form the  $B=4$  soliton, 9 of them (3 translations, 3 rotations and 3 isorotations) remain as zero modes of the new system. One might expect the remaining 15 modes would survive as low energy vibrational modes. This is one fewer than what we find: we have an additional breather mode.

The vibrational modes distort the  $B = 4$  soliton as illustrated in Figure 2, and explained in the Table. The modes naturally divide into two sets. The lower nine vibrational modes consist of deformations which, roughly, involve incompressible flow of the baryon charge. In contrast, the higher seven vibrational modes all have a ‘breathing’ character, in which local baryon charge expands or contracts to occupy a greater or lesser volume. The breather itself is simply a rescaling of the size of the soliton, with consequent change in density. The next mode up involves breathing motion of a dipole character, and the one above that of a quadrupole nature.

Remarkably, the vibrational modes below the breather fall into representations corresponding to those for small zero-mode deformations of the BPS 4-monopole solution [8]. The same phenomenon occurs in the deuteron [12], and it will be straightforward to check the  $B = 3, 5, 6, 7$  solitons using the methods described here. Qualitative similarities between Skyrme multisolitons and BPS multimono poles and their scattering dynamics have been noted before [9], [3]. Our finding suggests a connection between the lowest energy Skyrminion vibrational modes and the multimono pole moduli spaces. If it holds for higher nuclei, there will be  $4B - 7$  such modes. It would be very interesting to interpret this number in terms of individual nucleon degrees of freedom (presumably translations and spin/isospin).

We have investigated the effect of doubling the param-

eter  $\omega_\pi$  on the spectrum of modes. All modes move up in frequency. The nine lowest modes move up by 15-25 per cent, the breathing modes by 30-45 per cent, and the homogeneous pion modes roughly double in frequency. This has important phenomenological consequences. The lowest lying vibrational level for the real  $\alpha$  particle is at 20.1 MeV, whereas for  $\omega_\pi = .526$  our lowest lying frequency is at  $0.7\omega_\pi$ , or 94 MeV. For  $\omega_\pi = 1.052$  this is improved to 63 MeV. Whilst finite volume and grid spacing effects are present at a level of a few per cent in these results, we can safely conclude that a fit between the Skyrme model and nuclear gamma ray spectra will require parameters different than those usually used. It is interesting that a study of a single Skyrminion’s breather mode, attempting to identify it with the ‘Roper resonance’, also found that large values of  $\omega_\pi \sim 1.5$  were required [13]. It remains to be seen whether such large values of  $\omega_\pi$  can be accommodated in the theory.

We thank Richard Battye, Guy Moore, Conor Houghton, Paul Sutcliffe and especially Nick Manton for helpful discussions. We also acknowledge the Pittsburgh Supercomputing Center grant #AST9G3P for CRAY T3D supercomputer time.

- 
- [1] G. Adkins, C.R. Nappi, E. Witten, *Nuc. Phys.* **B228** (1983) 552.
  - [2] G. Adkins, C.R. Nappi, *Nuc. Phys.* **B233** (1984) 109.
  - [3] R. Battye and P. Sutcliffe, Imperial preprint 96-97/18, hep-th/9702089, *Phys. Rev. Lett.* submitted.
  - [4] E. Braaten, S. Townsend and L. Carson, *Phys. Lett.* **B235** (1990) 147.
  - [5] R.A. Leese, N.S. Manton, B.J. Schroers, *Nuc. Phys.* **B442** (1995) 228.
  - [6] N. Walet, *Nuc. Phys.* **A606** (1996) 429.
  - [7] R.A. Leese and N.S. Manton, *Nuc. Phys.* **A572** (1994) 575.
  - [8] N.S. Manton (private communication) noticed the correspondence between the representations of the proper cubic group we found with those for small deformations of the cubic 4-monopole BPS solution. One of us (NT) independently confirmed that the correspondence also holds for the parity assignments. See also C. J. Houghton, N.S. Manton and P.M. Sutcliffe, in preparation (1997).
  - [9] R. Battye and P. Sutcliffe, *Phys. Lett.* **391B** (1997) 150.
  - [10] For a review, see *Dynamics of the Standard Model*, J.F. Donoghue, E. Golowich and B.R. Holstein, Cambridge University Press (1992).
  - [11] M. Hamermesh, *Group Theory and its Application to Physical Problems*, Dover, 1962.
  - [12] C. Barnes, K. Baskerville and N. Turok, DAMTP preprint, in preparation (1997).
  - [13] J.D. Breit and C.R. Nappi, *Phys. Rev. Lett.* **53**, 889 (1984).

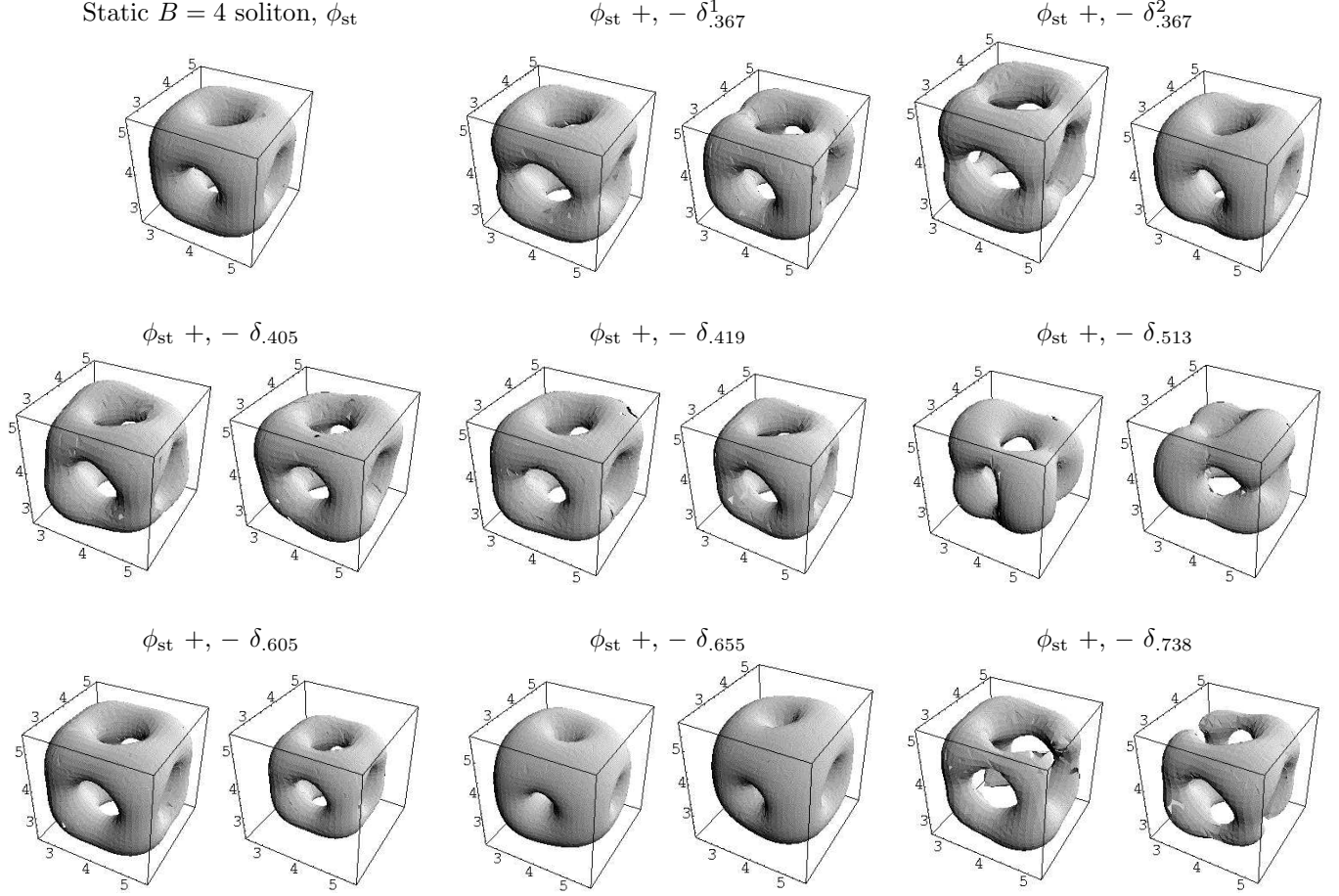


FIG. 2. Contours of constant baryon density for the  $B=4$  soliton, combined with its normal modes, as indexed by their frequencies in Figure 1. These modes were studied in a box of size  $8 \times 8 \times 8$ , with a grid spacing  $\Delta_x = \frac{1}{8}$  in Skyrme units. For comparison, the soliton itself is a cube roughly  $2 \times 2 \times 2$  in these units. For the case  $\omega = .367$  two orthogonal modes of the degenerate multiplet are shown, in other cases a single mode only is shown.

Frequency	Degeneracy	Symmetry	Description
.07	3	$F_1^+$	Rotations of the soliton. This is a zero mode broken by the finite box size.
.367	2	$E^+$	Lowest vibrational modes. One mode, $\delta^1$ , alternately pulls the $B = 4$ cube into two $B = 2$ donuts in two perpendicular directions. The orthogonal mode, $\delta^2$ , pulls it into four $B = 1$ edges one way and two $B = 2$ donuts the other.
.405	1	$A_2^-$	The corners of the cube make two interlacing tetrahedra. This mode pulls one tetrahedron out into four $B = 1$ corners, pushing the other one in.
.419	3	$F_2^+$	Deform two opposite faces of the cube into rhombuses.
.513	3	$F_2^-$	Deform the cube by pulling two opposite edges on one face, and the two perpendicular edges on the opposite face. This takes the cube to four $B = 1$ edges.
.545	2	$E^+$	Two of the pion $k = 0$ modes.
.587	1	$A_2^-$	The remaining pion $k = 0$ mode, with tetrahedral symmetry.
.605	1	$A_1^+$	The breathing mode, with the full cubic symmetry of the soliton.
.655	3	$F_2^-$	One face of the cube inflates, while the opposite face deflates.
.738	3	$F_2^+$	One pair of diagonally opposite edges inflates, the parallel pair deflates.
.908	3		The lowest nonzero ( $k = 1, 0, 0$ ) pion radiative mode.

TABLE I. Description of the Skyrme field  $B = 4$  normal modes marked in Figure 1. The notation of Hamermesh [11] is used for the representations of the cubic group  $O_h$ ; superscripts denote parity.

